Illustrating the Methodology in EKNR (2016): Some Simple Examples

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This note considers special cases of the model in Eaton, Kortum, Neiman, and Romalis (2016, henceforth EKNR). The objective is to demonstrate how the methodology works in simplified settings. Our web pages contain Matlab programs to implement the procedures described below.

The full model of EKNR includes multiple countries, multiple sectors, a complete set of country-specific input-output linkages, and factor shares that vary by country and sector. Producers in each sector produce heterogeneous goods with efficiency levels that differ across countries. Stocks of durables and structures are accumulated subject to adjustment costs. Capital is used in production (producer durables and structures) and by households (consumer durables and housing).

In the simpler setting considered here, a durables sector D produces a capital good and a services sector S produces a consumption good. There are no intermediates. Factor shares are identical across sectors. A single capital stock is used only for production.

We start with the very simplest case of a single country, treating producers within a sector as producing a homogenous good. Capital accumulation is not subject to adjustment costs. We then turn to a multi-country world, introducing differentiated goods and heterogeneous efficiencies. We also introduce costs of adjustment in capital accumulation, allowing us to ignore corner solutions of zero investment at the country level.

In both the closed economy and open economy settings, we use a planner's problem to characterize the solution to the competitive equilibrium and show how to compute it. In each case, we also show how to calibrate and compute the model in a more convenient way as in EKNR, extending to a dynamic setting the "exact hat algebra" approach used by Dekle, Eaton, and Kortum (2007) and extended in Costinot and Rodríguez-Clare (2014).

1 Closed Economy

We begin with the case of a single country. Sectors are denoted by $j \in \Omega = \{D, S\}$. At any date t the economy has a stock of durables K_t (the capital stock) and a supply of labor L_t. Capital services and labor are used to produce sectoral output in each $j \in \Omega$:

$$y_t^j = A_t^j B \left(L_t^j\right)^{\beta^L} \left(K_t^j\right)^{\beta^K},$$

with $\beta^L = 1 - \beta^K$. Here, A^j denotes total factor productivity. To cancel a term that would otherwise arise in the associated cost function, the constant is set to $B = \left(\beta^L\right)^{-\beta^L} \left(\beta^K\right)^{-\beta^K}$. Factor supplies are:

$$K_t = K_t^D + K_t^S \qquad \text{and} \qquad L_t = L_t^D + L_t^S.$$

The law of motion for the capital stock is:

$$K_{t+1} = \chi_t I_t + (1 - \delta) K_t, \tag{1}$$

where:

$$I_t = y_t^D \geq 0.$$

The utility of the representative household is:[1]

$$U = \sum_{t=0}^{\infty} \rho^t \ln C_t,$$

where $0 < \rho < 1$ and:

$$C_t = y_t^S.$$

1.1 Social Planner's Lagrangian

The social planner maximizes the utility of the representative household, subject to K_0 and the feasibility constraints. We can set up this problem as a Lagrangian:

$$\begin{aligned}
\mathcal{L} \;=\; & \sum_{t=0}^{\infty} \rho^t \Big[\ln C_t + \lambda_t^L \left(L_t - L_t^D - L_t^S\right) + \lambda_t^K \left(K_t - K_t^D - K_t^S\right) \\
& + \lambda_t^D \left(A_t^D B \left(L_t^D\right)^{\beta^L} \left(K_t^D\right)^{\beta^K} - I_t\right) + \lambda_t^S \left(A_t^S B \left(L_t^S\right)^{\beta^L} \left(K_t^S\right)^{\beta^K} - C_t\right) \\
& + \lambda_t^I I_t + \lambda_t^V \left(\chi_t I_t + (1 - \delta) K_t - K_{t+1}\right)\Big],
\end{aligned}$$

[1] The model in EKNR also has preference shocks, restricted to have no global component. Under that restriction, preference shocks drop out of the one-country model here.

where each λ is the Lagrange multiplier associated with the corresponding constraint. The transversality condition is:

$$\lim_{t\to\infty} \rho^t \lambda_t^V K_{t+1} = 0.$$

1.1.1 Input Demands

The first-order conditions for inputs of labor and capital to produce sector j output give us:

$$\lambda_t^j \beta^L \frac{y_t^j}{L_t^j} = \lambda_t^L,$$

and

$$\lambda_t^j \beta^K \frac{y_t^j}{K_t^j} = \lambda_t^K.$$

1.1.2 Shadow Values of Sectoral Output

Consider sector $j \in \Omega$. Multiplying the production function by the associated shadow value of output:

$$Y_t^j = \lambda_t^j y_t^j = \lambda_t^j A_t^j B \left(L_t^j\right)^{\beta^L} \left(K_t^j\right)^{\beta^K}.$$

Inserting the first-order conditions given above for inputs, we get:

$$Y_t^j = \lambda_t^j A_t^j B \left(\frac{\beta^L Y_t^j}{\lambda_t^L}\right)^{\beta^L} \left(\frac{\beta^K Y_t^j}{\lambda_t^K}\right)^{\beta^K}.$$

Constant returns to scale implies that Y_t^j cancels, giving us the shadow value of sector j output:

$$\lambda_t^j = \frac{b_t}{A_t^j}, \tag{2}$$

where the term:

$$b_t = \left(\lambda_t^L\right)^{\beta^L} \left(\lambda_t^K\right)^{\beta^K} \tag{3}$$

bundles the shadow costs of labor and capital in producing output.

1.1.3 Consumption and Investment

The first-order condition for C_t gives:

$$\lambda_t^S C_t = 1. \tag{4}$$

While consumption is as simple as can be, deriving the investment Euler equation requires several steps.

The first-order condition for K_t is:

$$\lambda_t^V = \rho\lambda_{t+1}^V(1-\delta) + \rho\lambda_{t+1}^K. \tag{5}$$

The first-order condition for I_t is:

$$\lambda_t^V\chi_t = \lambda_t^D - \lambda_t^I.$$

If $\lambda_t^I > 0$ then $I_t = 0$ and:

$$\lambda_t^V\chi_t < \lambda_t^D,$$

while if $I_t > 0$ then $\lambda_t^I = 0$ so that:

$$\lambda_t^V\chi_t = \lambda_t^D. \tag{6}$$

To keep the analysis as simple as possible, in what follows we will assume that exogenous terms and initial conditions guarantee strictly positive investment. In this case, combining (5) and (6) gives us the Euler equation:

$$\frac{\lambda_t^D}{\chi_t} = \rho\frac{\lambda_{t+1}^D}{\chi_{t+1}}(1-\delta) + \rho\lambda_{t+1}^K. \tag{7}$$

1.2 Solving the Competitive Equilibrium

Having derived the first order conditions to the planner's problem, we now reinterpret the shadow prices in the planner's problem as prices in a competitive equilibrium. In particular, we set $p_t^j = \lambda_t^j$ as the price of sector j output, $w_t = \lambda_t^L$ as the wage to labor, and $r_t = \lambda_t^K$ as the rental rate on capital. Choosing prices this way implies our numéraire is consumption expenditure:

$$p_t^S C_t = 1.$$

1.2.1 Static Relationships

For each sector $j \in \Omega$ in each country i, the value of output is:

$$Y_t^j = p_t^j y_t^j.$$

Since

$$Y_t^S = p_t^S C_t = 1,$$

GDP is:

$$Y_t = Y_t^D + 1.$$

Note that Y_t is not real GDP, but the value of GDP measured in terms of our numéraire.

Shadow values of factors are determined by the factor share equations for labor:

$$w_t L_t = \sum_{j \in \Omega} \beta^L Y_t^j = \beta^L Y_t,$$

and for capital:

$$r_t K_t = \beta^K Y_t.$$

The price of the capital good is:

$$p_t^D = \frac{1}{A_t^D} (w_t)^{\beta^L} (r_t)^{\beta^K} = \frac{Y_t}{B A_t^D (L_t)^{\beta^L} (K_t)^{\beta^K}}. \tag{8}$$

The price of the consumption good is closely related, since relative prices reflect relative efficiencies:[2]

$$p_t^S = \frac{A_t^D}{A_t^S} p_t^D = \frac{Y_t}{B A_t^S (L_t)^{\beta^L} (K_t)^{\beta^K}}.$$

1.2.2 Dynamics

We represent the dynamics with a pair of difference equations in the variables K (the capital stock) and Y (the value of GDP).[3]

We can derive one of the difference equations from the Euler equation (7), rewritten as:

$$\frac{p_t^D}{\chi_t} = \rho \frac{p_{t+1}^D}{\chi_{t+1}} (1-\delta) + \rho r_{t+1}. \tag{9}$$

Substituting the price expression (8) into the Euler equation:

$$\frac{Y_t}{\chi_t A_t^D (L_t)^{\beta^L} (K_t)^{\beta^K}} = \rho \left[\frac{1-\delta}{\chi_{t+1} A_{t+1}^D (L_{t+1})^{\beta^L} (K_{t+1})^{\beta^K - 1}} + B\beta^K \right] \frac{Y_{t+1}}{K_{t+1}}. \tag{10}$$

The other difference equation comes from the capital accumulation equation (1):

$$K_{t+1} = \chi_t \frac{Y_t^D}{p_t^D} + (1-\delta) K_t.$$

[2]For what we do here, we could have used a one-sector model by settting $A_t^S = A_t^D$. Retaining two sectors makes the analysis more like the multi-country case considered below, in which sectors play a critical role.

[3]It is more typical to express such a system in terms of K and C instead of K and Y. Either way works, since:

$$C_t = 1/p_t^S = B A_t^S (L_t)^{\beta^L} (K_t)^{\beta^K} / Y_t.$$

An advantage of Y is its close connection to r via $r_t K_t = \beta^K Y_t$ and to investment spending via $p_t^D I_t = Y_t^D = Y_t - 1$.

Substituting in (8) and rearranging:

$$K_{t+1} = \chi_t A_t^D B \left(L_t\right)^{\beta^L} \left(K_t\right)^{\beta^K} \frac{Y_t - 1}{Y_t} + (1-\delta)K_t. \tag{11}$$

We can solve equations (10) and (11) for the paths of Y and K. To do so, we also need to impose the transversality condition:

$$\lim_{t\to\infty} \rho^t \frac{p_t^D K_{t+1}}{\chi_t} = 0,$$

where we have used the fact that $\lambda_t^V = \lambda_t^D/\chi_t = p_t^D/\chi_t$.

We assume that the shocks eventually settle down to constant values, so that the system approaches a steady state. The initial value of K is determined by history. The initial value of Y is pinned down by the saddle path leading to this steady state. We now derive the value of the endogenous variables in the steady state.[4]

1.2.3 Steady State

Suppose after some date: $L_t = L$, $\chi_t = \chi$, $A_t^D = A^D$, and $A_t^S = A^S$. The economy will approach a steady state in which $K_t = K$.

In steady state, the Euler equation (10) reduces to:

$$\frac{1}{\chi A^D}\left(\frac{K}{L}\right)^{\beta^L} = \rho\frac{1-\delta}{\chi A^D}\left(\frac{K}{L}\right)^{\beta^L} + \rho B \beta^K.$$

Solving for K:

$$K = L\frac{\beta^K}{\beta^L}\left(\frac{\rho\chi A^D}{1-\rho\left(1-\delta\right)}\right)^{1/\beta^L}. \tag{12}$$

The steady-state capital stock is proportional to labor and increasing in the level of productivity in durables A^D and efficiency of investment χ.

The capital accumulation equation (11) evaluated at the steady state implies:

$$1 = \chi A^D B\left(\frac{K}{L}\right)^{-\beta^L}\frac{Y-1}{Y} + (1-\delta).$$

[4] Note that the shocks to productivity in durables and efficiency of investment enter only as a product in this system. That's all that matters for the key variables, although these shocks are separately identified by their effect on relative prices, see Justiniano, Primiceri, and Tambalotti (2011). The shock to productity in services doesn't show up at all in this system. This shock would be identified, however, if we looked at implications for real GDP.

Substituting in the steady-state Euler equation, we obtain:

$$\delta = \left(\frac{1-\rho\left(1-\delta\right)}{\rho\beta^K}\right)\frac{Y-1}{Y}. \tag{13}$$

In steady state, the share of investment spending in GDP is:

$$\frac{Y-1}{Y} = \frac{\delta\rho\beta^K}{1-\rho\left(1-\delta\right)}.$$

Capital income is:

$$rK = \beta^K\left(\frac{1-\rho\left(1-\delta\right)}{1-\rho\left(1-\delta\right)-\delta\rho\beta^K}\right),$$

GDP is:

$$Y = \frac{1-\rho\left(1-\delta\right)}{1-\rho\left(1-\delta\right)-\delta\rho\beta^K}, \tag{14}$$

and investment spending is:

$$Y^D = \frac{\delta\rho\beta^K}{1-\rho\left(1-\delta\right)-\delta\rho\beta^K}. \tag{15}$$

Unlike K, these four steady-state values are all invariant to L, A^D, or χ (recall that Y is not real GDP, but GDP measured in terms of our numéraire).

1.3 The EKNR Change Formulation

To compute the model as laid out so far, we require parameters (α, β^L, δ, ρ), an initial condition (K_0), and paths of exogenous shocks ($\{L_t\}$, $\left\{A_t^j\right\}$, and $\{\chi_t\}$). We now turn to an approach in which the computation of the competitive equilibrium is closely tied to a method of backing out paths of shocks from data. While this approach has a larger payoff in the multi-country setting, it is helpful to see it here in its simplest form.

Suppose we condition on Y_0, an endogenous variable that we can observe. To obtain the data analog to Y_0 in the model, we take the ratio of nominal GDP X_0 to nominal aggregate consumption spending X_0^C (both measured in some common currency):

$$Y_0 = \frac{X_0}{X_0^C}.$$

With this measure in hand, we will not need to know K_0, L_0, A_0^D, or χ_0 in order to determine the perfect foresight path of investment spending. (Implicitly, we are conditioning on the initial value of investment spending since $Y_0^D = Y_0 - 1$.) To demonstrate this result, we express the capital accumulation and Euler equations in changes (for a variable x, we define its *change* as $\hat{x}_{t+1} = x_{t+1}/x_t$).

1.3.1 Dynamic System in Changes

We can rewrite the capital accumulation equation (11) as:

$$\hat{K}_{t+1} - (1-\delta) = \chi_t A_t^D B \left(\frac{K_t}{L_t}\right)^{-\beta^L} \frac{Y_t - 1}{Y_t}.$$

Taking the ratio across periods, we get:

$$\frac{\hat{K}_{t+2} - (1-\delta)}{\hat{K}_{t+1} - (1-\delta)} = \hat{\chi}_{t+1} \hat{A}_{t+1}^D \left(\frac{\hat{K}_{t+1}}{\hat{L}_{t+1}}\right)^{-\beta^L} \frac{\left(Y_t \hat{Y}_{t+1} - 1\right)/(Y_t - 1)}{\hat{Y}_{t+1}}, \tag{16}$$

the capital accumulation equation in changes.[5]

We can rewrite the Euler equation (10) as:

$$1 = \rho \left[\frac{1-\delta}{\hat{\chi}_{t+1}\hat{A}_{t+1}^D} \left(\frac{\hat{K}_{t+1}}{\hat{L}_{t+1}}\right)^{\beta^L} + \chi_t A_t^D B \beta^K \left(\frac{K_t}{L_t}\right)^{-\beta^L}\right] \frac{\hat{Y}_{t+1}}{\hat{K}_{t+1}}.$$

Substituting in from the capital accumulation equation:

$$\chi_t A_t^D B \left(\frac{K_t}{L_t}\right)^{-\beta^L} = \left(\hat{K}_{t+1} - (1-\delta)\right) \frac{Y_t}{Y_t - 1},$$

we obtain:

$$1 = \rho \left[\frac{1-\delta}{\hat{\chi}_{t+1}\hat{A}_{t+1}^D} \left(\frac{\hat{K}_{t+1}}{\hat{L}_{t+1}}\right)^{\beta^L} + \beta^K \left(\hat{K}_{t+1} - (1-\delta)\right) \frac{Y_t}{Y_t - 1}\right] \frac{\hat{Y}_{t+1}}{\hat{K}_{t+1}}, \tag{18}$$

the Euler equation in changes.[6]

The capital accumulation equation in changes (16) together with the Euler equation in changes (18) form a system that, as we show in detail below, we can solve for paths of $\hat{K}_{t+1}$ and $\hat{Y}_{t+1}$,

[5] If we substitute the price expression (8) in changes:

$$\hat{p}_{t+1}^D = \frac{\hat{Y}_{t+1}}{\hat{A}_{t+1}^D \hat{K}_{t+1}} \left(\frac{\hat{K}_{t+1}}{\hat{L}_{t+1}}\right)^{\beta^L} \tag{17}$$

into (16), we get a special case of equation (24) in EKNR:

$$\hat{K}_{t+2} - (1-\delta) = \hat{\chi}_{t+1} \frac{\left(Y_t \hat{Y}_{t+1} - 1\right)/(Y_t - 1)}{\hat{p}_{t+1}^D \hat{K}_{t+1}} \left[\hat{K}_{t+1} - (1-\delta)\right].$$

[6] If we substitute the price expression in changes (17) into (18), we get a special case of equation (21) in EKNR:

$$\frac{1}{\rho} \frac{\hat{K}_{t+1}}{\hat{K}_{t+1} - (1-\delta)} = \frac{1}{\hat{\chi}_{t+1}} \frac{1-\delta}{\hat{K}_{t+1} - (1-\delta)} \hat{K}_{t+1} \hat{p}_{t+1}^D + \beta^K \frac{Y_t \hat{Y}_{t+1}}{Y_t - 1}.$$

given Y_0 and a sequence of changes in the shocks, which converge to constants.

1.3.2 Steady State in Changes

If, in the long run, the exogenous terms are $L_t = L$, $\chi_t = \chi$, and $A_t^D = A^D$ then $\hat{K}$ will converge to 1. To evaluate the steady state in changes, take the Euler equation in changes:

$$1 = \rho\left(1 - \delta + \beta^K \delta \frac{Y}{Y-1}\right)$$

or

$$\frac{Y}{Y-1} = \frac{1 - \rho(1-\delta)}{\rho \beta^K \delta},$$

which is the same condition as (13) in the levels version. In changes, we loose track of the steady state value of K, since the capital accumulation equation in changes reduces to $\hat{K} = 1$ in the steady state. It turns out that doesn't matter for our procedure.

1.3.3 Computation in Changes

An algorithm to compute the dynamic equilibrium begins with an observation of Y_0, the path for shocks $\left\{\hat{\chi}_{t+1}, \hat{A}_{t+1}^D, \hat{L}_{t+1}\right\}_{t=0}^{\infty}$ (eventually converging to 1's), and a guess of $\hat{K}_1$. Starting from $t = 0$:

1. From the Euler equation in changes (18), we solve for:

$$\hat{Y}_{t+1} = \frac{\hat{K}_{t+1}}{\rho\left(\frac{1-\delta}{\hat{\chi}_{t+1}\hat{A}_{t+1}^D}\left(\frac{\hat{K}_{t+1}}{\hat{L}_{t+1}}\right)^{\beta^L} + \beta^K\left(\hat{K}_{t+1} - (1-\delta)\right)\frac{Y_t}{Y_t - 1}\right)}.$$

2. From the capital accumulation equation (16), we update the capital stock:

$$\hat{K}_{t+2} = \hat{\chi}_{t+1}\hat{A}_{t+1}^D\left(\frac{\hat{K}_{t+1}}{\hat{L}_{t+1}}\right)^{-\beta^L}\frac{Y_t\hat{Y}_{t+1} - 1}{(Y_t - 1)\hat{Y}_{t+1}}\left[\hat{K}_{t+1} - (1-\delta)\right] + (1-\delta).$$

Proceeding in this manner, and setting $Y_{t+1} = Y_t\hat{Y}_{t+1}$, we adjust our guess of $\hat{K}_1$ until it produces a path that converges to the steady state. Thus, we are able to solve this system without needing to know the level of the capital stock or the level of any of the shocks.

We provide Matlab code to perform this computation, available on our web pages. In particular, the program there computes the solution to the competitive equilibrium and demonstrates that the same solution is obtained using this change formulation.

1.3.4 Backing out Shocks

While the computations above can be carried out with any shock paths (given that they converge to constants at some date in the future) we may want to consider shocks that are connected to the data that we have, or some perturbation of them. Suppose we have data through date T. We can then back out shocks through date T, assuming they remain constant after date T (when we lack data). In EKNR we back out shocks from data using the full model (see Sections 5 and 6 and Appendix C). We now describe how one would do so in this simplified setting.

To back out the shocks, we would first calculate $\hat{K}_{T+1}$ as the value consistent with a perfect foresight equilibrium at date T under the assumption that shocks are constant after date T. Next, given this value of $\hat{K}_{T+1}$ on the equilibrium path, we could use data to back out the $\hat{K}_{t+1}$ (for all $t < T$). Finally, with the path of $\hat{K}_{t+1}$ in hand, we could then back out the productivity and investment efficiency shocks as a composite, $\hat{\chi}_{t+1}\hat{A}^D_{t+1}$.

To carry out the first step, we would apply the algorithm described in Section 1.3.3 above for the problem beginning at date T, with all shocks constant going into the future.

For the second step, we would start with the version of the Euler equation in changes that matches equation (21) in EKNR:

$$\frac{\hat{K}_{t+1}}{\hat{K}_{t+1} - (1-\delta)} = \rho \frac{1}{\hat{\chi}_{t+1}} \frac{1-\delta}{\hat{K}_{t+1} - (1-\delta)} \hat{K}_{t+1} \hat{p}^D_{t+1} + \rho \beta^K \frac{Y_t \hat{Y}_{t+1}}{Y_t - 1}.$$

Substituting in the version of the capital accumulation equation in changes that matches equation (24) in EKNR:

$$\frac{1}{\hat{\chi}_{t+1}} \frac{\hat{p}^D_{t+1} \hat{K}_{t+1}}{\hat{K}_{t+1} - (1-\delta)} = \frac{\left(Y_t \hat{Y}_{t+1} - 1\right) / (Y_t - 1)}{\hat{K}_{t+2} - (1-\delta)},$$

we get:

$$\frac{\hat{K}_{t+1}}{\hat{K}_{t+1} - (1-\delta)} = \rho \frac{Y_{t+1} - 1}{Y_t - 1} \frac{(1-\delta)}{\hat{K}_{t+2} - (1-\delta)} + \rho \beta^K \frac{Y_{t+1}}{Y_t - 1},$$

which matches equation (26) in EKNR. Note that no shock values appear in this equation, allowing us to compute $\hat{K}_{t+1}$ given data (Y_t and Y_{t+1}) and a value for $\hat{K}_{t+2}$. We could thus use it to iterate backwards, given $\hat{K}_{T+1}$ from the first step.

Finally, with a path of changes in the capital stock in hand, we could proceed to the third step, using equation (16) to back out the shocks $\hat{\chi}_{t+1}\hat{A}^D_{t+1}$. The fact that these two shocks appear as a product does not matter since they enter as a product in calculation of the competitive equilibrium. Finally, we could obtain $\hat{L}_{t+1}$ directly from data on changes in employment.

2 International Trade

We now introduce an arbitrary number of countries, $n = 1, ..., \mathcal{N}$. Each country is similar to the closed economy except that we now introduce differentiated goods within sectors and heterogeneous production technologies across goods and countries. Countries may import and export individual durable capital goods subject to iceberg costs $d_{ni,t}$.

In particular, each sector is composed of a unit continuum of goods $z \in [0, 1]$, common across countries. Within each sector, goods are aggregated with a constant elasticity of substitution $\sigma \geq 0$.

The production function in country n for good z in sector $j \in \Omega$ is:

$$y_{n,t}^j(z) = a_{n,t}^j(z) B \left(L_{n,t}^j(z)\right)^{\beta^L} \left(K_{n,t}^j(z)\right)^{\beta^K},$$

where B is the same as for the closed economy. Production efficiency $a_{n,t}^j(z)$ is drawn from an extreme value distribution:

$$\Pr\left[a_{n,t}^j(z) \leq a\right] = e^{-T_{n,t}^j a^{-\theta}}, \tag{19}$$

where $\theta > \sigma - 1$.

Factors of production are constrained by:

$$K_{n,t} = \int_0^1 K_{n,t}^D(z)dz + \int_0^1 K_{n,t}^S(z)dz$$

and:

$$L_{n,t} = \int_0^1 L_{n,t}^D(z)dz + \int_0^1 L_{nt}^S(z)dz.$$

We assume that capital accumulates as:

$$K_{n,t+1} = \chi_{n,t} \left(I_{n,t}\right)^{\alpha} \left(K_{n,t}\right)^{1-\alpha} + (1-\delta)K_{nt}, \tag{20}$$

where $0 < \alpha < 1$ governs adjustment costs.[7]

To allow for shifts in relative spending across countries, we introduce shocks $\phi_{n,t}$ to preferences:

$$U_n = \sum_{t=0}^{\infty} \rho^t \phi_{n,t} \ln C_{n,t}.$$

[7]In the closed economy model described above, we set $\alpha = 1$. Here, we assume $\alpha < 1$ to avoid corner solutions in which investment in some country is zero.

2.1 Social Planner's Lagrangian

The world planner assigns a weight ω_n to country n's preferences. We restrict:

$$\sum_{n=1}^{\mathcal{N}} \omega_n \phi_{n,t} = 1,$$

so that preference shocks have no global component.

The planner's problem can be formulated in terms of the Lagrangian:

$$\begin{aligned}
\mathcal{L} \;=\; & \sum_{n=1}^{\mathcal{N}} \sum_{t=0}^{\infty} \rho^t \Bigg\{ \omega_n \phi_{n,t} \ln C_{n,t} + \lambda_{n,t}^{L} \left[L_{n,t} - \sum_{j \in \Omega} \int_0^1 L_{n,t}^{j}(z) dz \right] + \lambda_{n,t}^{K} \left[K_{n,t} - \sum_{j \in \Omega} \int_0^1 K_{n,t}^{j}(z) dz \right] \\
& + \sum_{j \in \Omega} \int_0^1 \lambda_{n,t}^{j}(z) \left[a_{n,t}^{j}(z) B \left(L_{n,t}^{j}(z) \right)^{\beta^L} \left(K_{n,t}^{j}(z) \right)^{\beta^K} - y_{n,t}^{j}(z) \right] dz \\
& + \lambda_{n,t}^{S} \left[\left(\int_0^1 y_{n,t}^{S}(z)^{(\sigma-1)/\sigma} dz \right)^{\sigma/(\sigma-1)} - C_{n,t} \right] + \lambda_{n,t}^{D} \left[\left(\int_0^1 x_{n,t}^{D}(z)^{(\sigma-1)/\sigma} dz \right)^{\sigma/(\sigma-1)} - I_{n,t} \right] \\
& + \int_0^1 \hat{\lambda}_{n,t}^{D}(z) \left[y_{n,t}^{D}(z) - \sum_{m=1}^{\mathcal{N}} d_{mn,t} x_{mn,t}^{D}(z) \right] dz + \int_0^1 \tilde{\lambda}_{n,t}^{D}(z) \left[\sum_{i=1}^{\mathcal{N}} x_{ni,t}^{D}(z) - x_{n,t}^{D}(z) \right] dz \\
& + \sum_{i=1}^{\mathcal{N}} \int_0^1 \tilde{\lambda}_{ni,t}^{D}(z) x_{ni,t}^{D}(z) dz + \lambda_{n,t}^{V} \left[\chi_{n,t} \left(I_{n,t} \right)^{\alpha} \left(K_{n,t} \right)^{1-\alpha} + (1-\delta) K_{n,t} - K_{n,t+1} \right] \Bigg\},
\end{aligned}$$

where each λ is the Lagrange multiplier associated with the corresponding constraint. Initial capital stocks $K_{n,0}$ are given. The transversality conditions are:

$$\lim_{t \to \infty} \rho^t \lambda_{n,t}^{V} K_{n,t+1} = 0,$$

for each $n = 1, 2, ..., \mathcal{N}$.

2.1.1 The Shadow Cost of Production

Following the same argument that led to equation (2), the shadow cost of producing good z in sector j in country n is:

$$\lambda_{n,t}^{j}(z) = \frac{b_{n,t}}{a_{n,t}^{j}(z)}. \tag{21}$$

The shadow cost of a bundle of factors in n is, in parallel to (3), $b_{n,t} = \left(\lambda_{n,t}^{L} \right)^{\beta^L} \left(\lambda_{n,t}^{K} \right)^{\beta^K}$.

2.1.2 The Shadow Value of the Consumption

The first-order condition for $y^S_{n,t}(z)$ gives:

$$\lambda^S_{n,t}(z) = \lambda^S_{n,t} \left(\int_0^1 y^S_{n,t}(z')^{(\sigma-1)/\sigma} dz' \right)^{1/(\sigma-1)} y^S_{n,t}(z)^{-1/\sigma}.$$

Substituting in aggregate consumption and rearranging:

$$\frac{Y^S_{n,t}(z)}{\lambda^S_{n,t} C_{n,t}} = \left(\frac{\lambda^S_{n,t}(z)}{\lambda^S_{n,t}} \right)^{-(\sigma-1)}. \tag{22}$$

Integrating both sides of (22), and using the fact that:[8]

$$\int_0^1 Y^S_{n,t}(z) dz = \lambda^S_{n,t} C_{n,t}, \tag{23}$$

we get:

$$\left(\lambda^S_{n,t}\right)^{-(\sigma-1)} = \int_0^1 \left(\lambda^S_{n,t}(z)\right)^{-(\sigma-1)} dz.$$

Based on (19) and (21), we can treat $\lambda^S_{n,t}(z)$ as a random variable with distribution:

$$G^S_{n,t}(x) = \Pr\left[\lambda^S_{n,t}(z) \leq x\right] = 1 - e^{-T^S_{n,t} b^{-\theta}_{n,t} x^\theta}.$$

We can therefore perform the integration above as:

$$\left(\lambda^S_{n,t}\right)^{-(\sigma-1)} = \int_0^\infty x^{-(\sigma-1)} dG^S_{n,t}(x) = \left[\gamma \left(T^S_{n,t}\right)^{-1/\theta} b_{n,t}\right]^{-(\sigma-1)},$$

where γ is related to the gamma function:

$$\gamma = \left[\Gamma\left(\frac{\theta - \sigma + 1}{\theta}\right)\right]^{-1/(\sigma-1)}.$$

[8] While seemingly obvious, it takes a bit of work to derive (23). Start with:

$$\left(\int_0^1 \left(y^S_{n,t}(z)\right)^{(\sigma-1)/\sigma} dz \right)^{\sigma/(\sigma-1)} = C_{n,t}$$

and multiply both sides by $\left(\lambda^S_{n,t}\right)^{1/(\sigma-1)}$, which can be brought inside the integral. Inside the integral, from (22), substitute in:

$$\left(\lambda^S_{n,t}\right)^{1/(\sigma-1)} = \left(\frac{Y^S_{n,t}(z)}{C_{n,t}} \right)^{1/(\sigma-1)} \left(\frac{\lambda^S_{n,t}(z)}{\lambda^S_{n,t}} \right)$$

and simplify.

Defining sectoral productivity as:

$$A_{n,t}^{j} = (1/\gamma)\left(T_{n,t}^{j}\right)^{1/\theta},$$

for $j \in \Omega$, we have a simple and intuitive expression for the shadow value of consumption:

$$\lambda_{n,t}^{S} = \frac{b_{n,t}}{A_{n,t}^{S}},$$

the analog of equation (2) for the closed economy.

2.1.3 Specialization and the Shadow Value of the Investment

The first order condition for $x_{ni,t}^{D}(z)$ is:

$$\tilde{\lambda}_{n,t}^{D}(z) + \tilde{\lambda}_{ni,t}^{D}(z) = \hat{\lambda}_{i,t}^{D}(z) d_{ni,t}.$$

If $\tilde{\lambda}_{ni,t}^{D}(z) > 0$ then:

$$\tilde{\lambda}_{n,t}^{D}(z) < \hat{\lambda}_{i,t}^{D}(z) d_{ni,t}$$

and $x_{ni,t}^{D}(z) = 0$, while if $x_{ni,t}^{D}(z) > 0$ then $\tilde{\lambda}_{ni,t}^{D}(z) = 0$ and:

$$\tilde{\lambda}_{n,t}^{D}(z) = \hat{\lambda}_{i,t}^{D}(z) d_{ni,t}.$$

Since country n will obtain capital good z from somewhere:

$$\tilde{\lambda}_{n,t}^{D}(z) = \min_{i}\left\{\hat{\lambda}_{i,t}^{D}(z) d_{ni,t}\right\}. \tag{24}$$

The first order condition with respect to $y_{n,t}^{D}(z)$ is simply:

$$\hat{\lambda}_{n,t}^{D}(z) = \lambda_{n,t}^{D}(z),$$

which, in combination with (21), lets us rewrite (24) as:

$$\tilde{\lambda}_{n,t}^{D}(z) = \min_{i}\left\{\frac{b_{i,t}}{a_{i,t}^{D}(z)} d_{ni,t}\right\}. \tag{25}$$

Based on (19) and (25), we can treat $\tilde{\lambda}^D_{n,t}(z)$ as a random variable with distribution:

$$
\begin{aligned}
G^D_{n,t}(x) &= \Pr\left[\tilde{\lambda}^D_{n,t}(z) \leq x\right] = 1 - \prod_{i=1}^{\mathcal{N}} \Pr\left[\frac{b_{i,t}}{a^D_{i,t}(z)} d_{ni,t} \geq x\right] \\
&= 1 - \prod_{i=1}^{\mathcal{N}} \Pr\left[a^D_{i,t}(z) \leq \frac{b_{i,t}}{x} d_{ni,t}\right] = 1 - \prod_{i=1}^{\mathcal{N}} e^{-T^D_{i,t}(b_{i,t}d_{ni,t})^{-\theta} x^{\theta}} \\
&= 1 - e^{-\Phi_{n,t} x^{\theta}},
\end{aligned}
$$

where:

$$
\Phi_{n,t} = \sum_{i=1}^{\mathcal{N}} T^D_{i,t} \left(b_{i,t} d_{ni,t}\right)^{-\theta}.
$$

The first-order condition for $x^D_{n,t}(z)$ gives us the analog of (22), now for durables:

$$
\frac{X^D_{n,t}(z)}{\lambda^D_{n,t} I_{n,t}} = \left(\frac{\tilde{\lambda}^D_{n,t}(z)}{\lambda^D_{n,t}}\right)^{-(\sigma-1)}.
$$

Integrating both sides we get:

$$
\left(\lambda^D_{n,t}\right)^{-(\sigma-1)} = \int_0^{\infty} x^{-(\sigma-1)} dG^D_{n,t}(x) = \left[\gamma \left(\Phi_{n,t}\right)^{-1/\theta}\right]^{-(\sigma-1)}.
$$

The shadow value of investment is thus:

$$
\lambda^D_{n,t} = \left[\sum_{i=1}^{\mathcal{N}} \left(\frac{b_{i,t} d_{ni,t}}{A^D_{i,t}}\right)^{-\theta}\right]^{-1/\theta}.
$$

The fraction of durable goods that country n obtains as imports from i is:

$$
\pi_{ni,t} = \left(\frac{b_{i,t} d_{ni,t}}{\lambda^D_{n,t} A^D_{i,t}}\right)^{-\theta},
$$

so that the value of durable good production in country i is:

$$
Y^D_{i,t} = \sum_{n=1}^{\mathcal{N}} \pi_{ni,t} X^D_{n,t},
$$

where

$$
X^D_{n,t} = \lambda^D_{n,t} I_{n,t} \geq 0.
$$

2.1.4 Consumption and Investment

The first-order condition for $C_{n,t}$ gives:

$$\lambda_{n,t}^S C_{n,t} = \omega_n \phi_{n,t}.$$

The first-order condition for $I_{n,t}$ is:

$$\lambda_{n,t}^V = \frac{\lambda_{n,t}^D}{\alpha \chi_{n,t}} \left(\frac{I_{n,t}}{K_{n,t}}\right)^{1-\alpha},$$

while the first-order condition for $K_{n,t}$ is:

$$\lambda_{n,t}^V = \rho \lambda_{n,t+1}^V \left[\chi_{n,t+1} (1-\alpha) \left(\frac{I_{n,t+1}}{K_{n,t+1}}\right)^{\alpha} + (1-\delta)\right] + \rho \lambda_{n,t+1}^K.$$

Combining the two gives us the Euler equation for country n:

$$\frac{\lambda_{n,t}^D}{\alpha \chi_{n,t}} \left(\frac{I_{n,t}}{K_{n,t}}\right)^{1-\alpha} = \rho \frac{\lambda_{n,t+1}^D}{\alpha \chi_{n,t+1}} \left(\frac{I_{n,t+1}}{K_{n,t+1}}\right)^{1-\alpha} \left[\chi_{n,t+1} (1-\alpha) \left(\frac{I_{n,t+1}}{K_{n,t+1}}\right)^{\alpha} + (1-\delta)\right] + \rho \lambda_{n,t+1}^K.$$

2.2 Computing the Competitive Equilibrium

Replacing the relevant Lagrange multipliers with the corresponding competitive prices, we let $p_{n,t}^j = \lambda_{n,t}^j$, $w_{n,t} = \lambda_{n,t}^L$, and $r_{n,t} = \lambda_{n,t}^K$. Choosing prices this way implies our numéraire is world output of the services sector (or world consumption expenditure):

$$\sum_{n=1}^{\mathcal{N}} Y_{n,t}^S = \sum_{n=1}^{\mathcal{N}} p_{n,t}^S C_{n,t} = \sum_{n=1}^{\mathcal{N}} \omega_n \phi_{n,t} = 1.$$

We now proceed to list the equations needed to calculate a competitive equilibrium, given the exogenous terms (for each country n, sector j, and date t when necessary): α, β^L, δ, σ, θ, ρ, ω_n, $\{K_{n,0}\}$, $\{L_{n,t}\}$, $\left\{A_{n,t}^j\right\}$, $\{\phi_{n,t}\}$, $\{\chi_{n,t}\}$, and $\{d_{ni,t}\}$.

We start with country n's GDP, which is defined as:

$$Y_{n,t} = Y_{n,t}^D + Y_{n,t}^S.$$

For expository purposes, pretend that we know the paths of GDP $\{Y_{n,t}\}$. We can then show how everything else can be written in terms of these GDP paths, before showing how they are themselves nailed down.

Given GDP, we get the wage, $w_{n,t} = \beta^L Y_{n,t}/L_{n,t}$, the rental rate, $r_{n,t} = \beta^K Y_{n,t}/K_{n,t}$, and hence the price of a bundle of factors, $b_{n,t} = (w_{n,t})^{\beta^L} (r_{n,t})^{\beta^K}$. The price of the consumption good is:[9]

$$p_{n,t}^{S} = \frac{b_{n,t}}{A_{n,t}^{S}} = \frac{(w_{n,t})^{\beta^L} (r_{n,t})^{\beta^K}}{A_{n,t}^{S}}.$$

The price of the capital good depends on costs of production in all countries:

$$p_{nt}^{D} = \left(\sum_{i=1}^{\mathcal{N}} \left(\frac{b_{i,t} d_{ni,t}}{A_{i,t}^{D}} \right)^{-\theta} \right)^{-1/\theta}.$$

Given those prices, we have bilateral trade shares:

$$\pi_{ni,t} = \left(\frac{b_{i,t} d_{ni,t}}{p_{n,t}^{D} A_{i,t}^{D}} \right)^{-\theta}.$$

The sectoral composition of GDP is given by :

$$Y_{n,t}^{D} = Y_{n,t} - Y_{n,t}^{S} = Y_{n,t} - \omega_n \phi_{n,t}.$$

We know that $Y_{n,t}^{D} > 0$ since there will always be draws from the extreme value distribution that will lead any country to produce some positive measure of capital goods. Equilibrium paths for GDP guarantee that expenditures on investment satisfy the Euler equation:

$$\frac{p_{n,t}^{D}}{\alpha \chi_{n,t}} \left(\frac{X_{n,t}^{D}}{p_{n,t}^{D} K_{n,t}} \right)^{1-\alpha} = \rho \frac{p_{n,t+1}^{D}}{\alpha \chi_{n,t+1}} \left(\frac{X_{n,t+1}^{D}}{p_{n,t+1}^{D} K_{n,t+1}} \right)^{1-\alpha} \left[\chi_{n,t+1} (1-\alpha) \left(\frac{X_{n,t+1}^{D}}{p_{n,t+1}^{D} K_{n,t+1}} \right)^{\alpha} + (1-\delta) \right] + \rho r_{n,t+1}$$

and also satisfy the trade equation (given trade shares and the value of production):

$$Y_{i,t}^{D} = \sum_{n=1}^{\mathcal{N}} \pi_{ni,t} X_{n,t}^{D},$$

which implies:

$$\sum_{i=1}^{\mathcal{N}} Y_{i,t}^{D} = \sum_{n=1}^{\mathcal{N}} X_{n,t}^{D}.$$

We can update the capital stock as:

$$K_{n,t+1} = \chi_{n,t} \left(\frac{X_{n,t}^{D}}{p_{n,t}^{D}} \right)^{\alpha} (K_{n,t})^{1-\alpha} + (1-\delta) K_{nt},$$

and proceed as above for date $t+1$. To nail down the initial values of GDP, we need to impose

[9]It turns out that the price of the consumption good doesn't matter for investment spending. Thus, in what follows, we can ignore shocks to productivity in the service sector.

the transversality conditions:

$$\lim_{t\to\infty} \rho^t \frac{p^D_{n,t} K_{n,t+1}}{\alpha \chi_{n,t}} \left(\frac{X^D_{n,t}}{p^D_{n,t} K_{n,t}} \right)^{1-\alpha} = 0.$$

We will also impose that our shock paths all converge to constants, so that the system has a well-defined steady state, as we describe below.

2.2.1 Steady State

We assume that after some date t: $L_{n,t} = L_n$, $\chi_{n,t} = \chi_n$, $A^D_{n,t} = A^D_n$, and $\phi_{n,t} = \phi_n$. The economy will approach a steady state in which $K_{n,t} = K_n$. Note that investment is strictly positive in a steady state, since it must counteract depreciation.

The steady-state price of capital goods is:

$$p^D_n = \left(\sum_{i=1}^{\mathcal{N}} \left(\frac{b_i d_{ni}}{A^D_i} \right)^{-\theta} \right)^{-1/\theta}$$

and trade shares are:

$$\pi_{ni} = \left(\frac{b_i d_{ni}}{p^D_n A^D_i} \right)^{-\theta},$$

where the cost of a bundle of factors is:

$$b_n = \frac{Y_n}{B \left(L_n\right)^{\beta^L} \left(K_n\right)^{\beta^K}}.$$

In steady state the capital accumulation equation reduces to:

$$\frac{X^D_n}{p^D_n K_n} = \left(\frac{\delta}{\chi_n} \right)^{1/\alpha}.$$

Substituting into the Euler equation in steady state:

$$\frac{p^D_n}{\alpha \chi_n} \left(\frac{\delta}{\chi_n} \right)^{(1/\alpha)-1} = \rho \frac{p^D_n}{\alpha \chi_n} \left(\frac{\delta}{\chi_n} \right)^{(1/\alpha)-1} (1-\alpha\delta) + \rho \frac{\beta^K Y_n}{K_n},$$

which simplifies to:

$$\frac{X^D_n}{Y_n} = \beta^K \frac{\alpha\delta\rho}{1-\rho(1-\alpha\delta)}. \tag{26}$$

For the world as a whole, we have:

$$Y = 1 + \sum_{i=1}^{\mathcal{N}} Y^D_i = 1 + X^D = 1 + \beta^K \frac{\alpha\delta\rho}{1-\rho(1-\alpha\delta)} Y,$$

so that:

$$Y = \frac{1 - \rho(1 - \alpha\delta)}{1 - \rho(1 - \alpha\delta) - \alpha\delta\rho\beta^K},$$

and:

$$X^D = Y^D = \frac{\alpha\delta\rho\beta^K}{1 - \rho(1 - \alpha\delta) - \alpha\delta\rho\beta^K}.$$

For $\alpha = 1$ these expression reduce to (14) and (15), as we obtained in the one-country case.[10]

2.3 The EKNR Change Formulation

Suppose we condition on observations of GDP, $Y_{n,0}$, and bilateral trade shares, $\pi_{ni,0}$. To measure $Y_{n,0}$ we take the ratio of GDP $X_{n,0}$ measured in dollars to nominal world consumption spending X_0^C measured in dollars:

$$Y_{n,0} = \frac{X_{n,0}}{X_0^C}.$$

With this measure in hand, we will not need to know $K_{n,0}$, L_{n0}, $A_{n,0}^D$, $\chi_{n,0}$, or $d_{ni,0}$ in order to determine the perfect foresight path of investment spending. To demonstrate this result, we express the system in terms of variables representing the ratio of values in $t+1$ to t: i.e. $\hat{x}_{t+1} = x_{t+1}/x_t$.

2.3.1 Dynamic System in Changes

We can rewrite the bilateral trade equation in changes as:

$$\hat{\pi}_{ni,t+1} = \left(\frac{\hat{b}_{i,t+1}\hat{d}_{ni,t+1}}{\hat{p}_{n,t+1}^D \hat{A}_{i,t+1}^D} \right)^{-\theta},$$

where $\hat{b}_{i,t+1} = (\hat{w}_{i,t+1})^{\beta^L} (\hat{r}_{i,t+1})^{\beta^K}$, $\hat{w}_{i,t+1} = \hat{Y}_{i,t+1}/\hat{L}_{i,t+1}$, and $\hat{r}_{i,t+1} = \hat{Y}_{i,t+1}/\hat{K}_{i,t+1}$. The price of capital goods in changes is:

$$\hat{p}_{n,t+1}^D = \left[\sum_{i=1}^{\mathcal{N}} \pi_{ni,t} \left(\frac{\hat{b}_{i,t+1}\hat{d}_{ni,t+1}}{\hat{A}_{i,t+1}^D} \right)^{-\theta} \right]^{-1/\theta}.$$

The change in consumption spending is simply:

$$\hat{X}_{n,t+1}^C = \hat{\phi}_{n,t+1}.$$

[10] Because of the non-linearity in the bilateral trade equations, we cannot go further in solving the steady state in closed form.

The Euler equation in changes is:

$$\frac{1}{\rho}\frac{\hat{K}_{n,t+1}}{\hat{K}_{n,t+1}-(1-\delta)} = \hat{X}^D_{n,t+1}\left[(1-\alpha)+\frac{1}{\hat{\chi}_{n,t+1}}\left(\frac{\hat{K}_{n,t+1}\hat{p}^D_{n,t+1}}{\hat{X}^D_{n,t+1}}\right)^{\alpha}\frac{1-\delta}{\hat{K}_{n,t+1}-(1-\delta)}\right]+\alpha\beta^K\frac{Y_{n,t}\hat{Y}_{n,t+1}}{X^D_{n,t}}$$

and the condition that a country's production is absorbed globally, written in changes, is:

$$Y^D_{i,t}\hat{Y}^D_{i,t+1} = \sum_{n=1}^{\mathcal{N}}\pi_{ni,t+1}X^D_{n,t}\hat{X}^D_{n,t+1}.$$

Finally, we can write the capital accumulation equation in changes as:

$$\hat{K}_{n,t+2}-(1-\delta) = \hat{\chi}_{n,t+1}\left(\frac{\hat{X}^D_{n,t+1}}{\hat{p}^D_{n,t+1}\hat{K}_{n,t+1}}\right)^{\alpha}\left[\hat{K}_{n,t+1}-(1-\delta)\right].$$

We provide Matlab code on our web pages to solve this system. The program computes the solution to the competitive equilibrium in the conventional way (in levels) and demonstrates that the same solution is obtained using the change formulation as described here.

A major advantage of the change formulation in this multi-country case is the ease with which we can back out shocks from data, in particular the shocks to trade costs, $\hat{d}_{ni,t}$. The basic strategy is similar to that described in Section 1.3.4 above. For more details on backing out shocks, see Section 5 of EKNR. For more details on computation, see Appendix C of EKNR.

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